Math. Z. 248, 147–172 (2004) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-004-0657-9

Homotopy Self-Equivalences of 4-Manifolds

Ian Hambleton[1], **Matthias Kreck**[2]

[1] Department of Mathematics & Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada (e-mail: ian@math.mcmaster.ca)
[2] Mathematisches Institut, Universität Heidelberg, D-69120, Heidelberg, Germany (e-mail: kreck@mathi.uni-heidelberg.de)

Received: 16 May 2003; in final form: 19 November 2003 /
Published online: 9 April 2004 –

Abstract. We establish a braid of interlocking exact sequences containing the group of homotopy self-equivalences of a smooth or topological 4-manifold. The braid is computed for manifolds whose fundamental group is finite of odd order.

1 Introduction

Let M^4 be a closed, oriented, smooth or topological 4-manifold. We wish to study the group $\mathrm{Aut}(M)$ of homotopy classes of homotopy self-equivalences $f\colon M \to M$, using techniques from surgery and bordism theory. We will always assume that M is connected. Here is an overview of our results, starting with an informal description of some related objects.

The group $\mathcal{H}(M)$ consists of oriented h-cobordisms W^5 from M to M, under the equivalence relation induced by h-cobordism relative to the boundary. The orientation of W induces opposite orientations on the two boundary components M. An h-cobordism gives a homotopy self-equivalence of M, and we get a homomorphism $\mathcal{H}(M) \to \mathrm{Aut}(M)$.

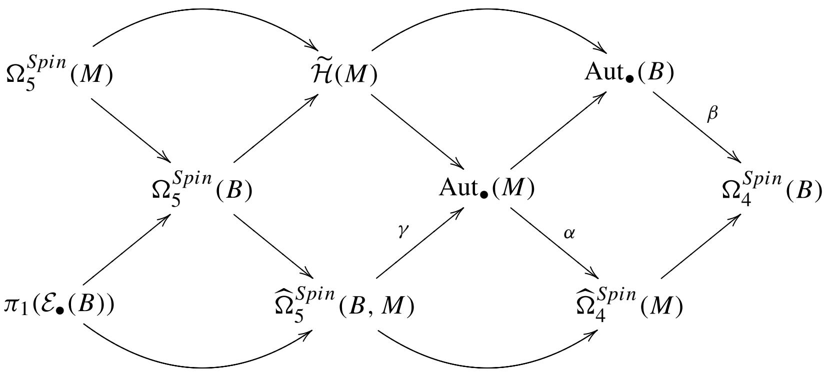
Let $B = B(M)$ denote the 2-type of M. It is a fibration over $K(\pi_1(M), 1)$, with fibre $\pi_2(M)$ determined by a k-invariant $k_M \in H^3(\pi_1; \pi_2)$, obtained from M by attaching cells of dimension ≥ 4 to kill the homotopy groups in dimensions ≥ 3. The natural map $c\colon M \to B$ is 3-connected, and we refer to this as the classifying map of M. There is an induced homomorphism $\mathrm{Aut}(M) \to \mathrm{Aut}(B)$, the group of homotopy classes of homotopy self-equivalences of B, by obstruction theory and the naturality of the construction. If M is a spin manifold, we will also be using the smooth (or topological) bordism groups $\Omega_n^{Spin}(B)$. By imposing the requirement that the reference maps to M must have degree zero, we obtain modified bordism

Research partially supported by NSERC Research Grant A4000. The authors thank the Max Planck Institut für Mathematik in Bonn, Germany, for hospitality and support.

groups $\widehat{\Omega}_4^{Spin}(M)$ and $\widehat{\Omega}_5^{Spin}(B, M)$. When $w_2(M) \neq 0$, we will use the appropriate bordism groups of the *normal* 2-*type*. See [18], [12], [13] for this theory.

A variation of $\mathcal{H}(M)$, denoted $\widetilde{\mathcal{H}}(M)$, will also be useful. This is the group of oriented bordisms $(W, \partial_- W, \partial_+ W)$ with $\partial_\pm W = M$, equipped with a map $F\colon W \to M$. We require the restrictions $F|_{\partial_\pm W}$ to the boundary components to be homotopy equivalences (and the identity on the component $\partial_- W$). The equivalence relation on these objects is induced by bordism (extending the map to M) relative to the boundary (see Section 2.2 for the details).

Our strategy is to compare $\mathrm{Aut}(M)$ to these other groups by means of various interlocking exact sequences. For technical reasons, we will restrict ourselves to homotopy self-equivalences preserving both the given orientation on M and a fixed base-point $x_0 \in M$. Let $\mathrm{Aut}_\bullet(M)$ denote the group of homotopy classes of such homotopy self-equivalences. We will also define "pointed" versions of the other objects, including the space $\mathcal{E}_\bullet(B)$ of base-point preserving homotopy equivalences of B, and the group $\mathrm{Aut}_\bullet(B) = \pi_0(\mathcal{E}_\bullet(B))$. Our main qualitative result in the spin case, Theorem 2.16, is expressed in a commutative braid



of exact sequences, valid for any closed, oriented smooth or topological spin 4-manifold M (see Theorem 3.15 for the analogous statement in the non-spin case). The maps labelled α and β are not homomorphisms, so exactness is understood in the sense of "pointed sets" (meaning that $image = kernel$, where $kernel$ is the pre-image of the base point).

For the special case when the fundamental group $\pi_1(M, x_0)$ is finite of odd order, we can compute this braid to obtain an explicit formula for $\mathrm{Aut}_\bullet(M)$ and a description of $\mathcal{H}(M)$. The simply-connected case was already known (see [4], [22],[16], [14]), but our proof is new even in that case.

In [9, p. 85] we defined the *quadratic* 2-*type* of M as the 4-tuple $[\pi_1, \pi_2, k_M, s_M]$, where s_M is the intersection form on $\pi_2(M)$. The isometries of the quadratic 2-type consist of all pairs of isomorphisms $\chi\colon \pi_1(M, x_0) \to \pi_1(M, x_0)$ and $\phi\colon \pi_2(M) \to \pi_2(M)$, such that $\phi(gx) = \chi(g)\phi(x)$, which preserve the k-invariant and the intersection form. The group $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$ is thus a subgroup of the arithmetic group $SO(\pi_2(M), s_M)$.

Theorem A. *Let M^4 be a connected, closed, oriented smooth (or topological) manifold of dimension* 4. *If $\pi_1(M, x_0)$ has odd order, then*

$$\mathrm{Aut}_\bullet(M) \cong KH_2(M; \mathbf{Z}/2) \rtimes \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$$

where $KH_2(M; \mathbf{Z}/2) := \ker(w_2\colon H_2(M; \mathbf{Z}/2) \to \mathbf{Z}/2))$.

The image of $\widetilde{\mathcal{H}}(M)$ in $\mathrm{Aut}_\bullet(M)$ is isomorphic to $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$, giving the semi-direct product splitting, and the action on the normal subgroup $KH_2(M; \mathbf{Z}/2)$ by $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$ is induced by the action of $\mathrm{Aut}_\bullet(M)$ on homology.

Let $\mathcal{S}^h(M \times I, \partial)$ denote the structure group of smooth or topological manifold structures on $M \times I$, relative to the given structure on $\partial(M \times I)$. Let $\tilde{L}_6(\mathbf{Z}[\pi])$ denote the *reduced* Wall group (see [22, Chap. 9]) defined as the cokernel of the split injection $L_6(\mathbf{Z}) \to L_n(\mathbf{Z}[\pi])$ induced by the inclusion $1 \to \pi$ of the trivial group. The group $\mathcal{H}(M)$ of smooth or topological h-cobordisms from M to M is now determined up to extensions.

Theorem B. *Let M^4 be a connected, closed, oriented smooth (or topological) manifold of dimension* 4. *If $\pi_1(M, x_0)$ has odd order, then there is a short exact sequence of groups:*

$$1 \to \mathcal{S}^h(M \times I, \partial) \to \mathcal{H}(M) \to \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]) \to 1$$

where the normal subgroup $\mathcal{S}^h(M \times I, \partial)$ is abelian and is determined up to extension by the short exact sequence

$$0 \to \tilde{L}_6(\mathbf{Z}[\pi_1(M, x_0)]) \to \mathcal{S}^h(M \times I, \partial) \to H_1(M; \mathbf{Z}) \to 0$$

of groups and homomorphisms.

In the simply-connected case, $\mathcal{S}^h(M \times I, \partial) = 0$ so the group of h-cobordisms is just isomorphic to the isometries of the intersection form of M.

The authors would like to thank the referee for useful comments on the first version of this paper.

2 Spin bordism groups and exact sequences

We now define more precisely the objects and maps which appear in our braid, beginning with the case when M is a spin 4-manifold. We fix a lift $\nu_M : M \to BSpin$ of the classifying map for the stable normal bundle of M. Let $\Omega_*^{Spin}(M)$ or $\Omega_*^{Spin}(B)$ denote the singular bordism groups of topological spin manifolds equipped with a reference map to M or B. The discussion below holds for smooth bordism (when M is a smooth 4-manifold) without any essential changes.

2.1 The map α

We fix a base-point $x_0 \in M$ and the corresponding base-point in B, thinking of B as constructed from M by adding cells of dimension ≥ 4. The evaluation map at x_0 gives a fibration

$$\mathcal{E}_\bullet(M) \to \mathcal{E}(M) \to M$$

where $\mathcal{E}(M)$ denotes the space of orientation-preserving homotopy self-equivalences of M. We have a long exact sequence

$$\cdots \to \pi_1(\mathcal{E}(M)) \to \pi_1(M, x_0) \to \pi_0(\mathcal{E}_\bullet(M)) \to \pi_0(\mathcal{E}(M)) \to \pi_0(M) .$$

The image $G(M, x_0) := \mathrm{Im}(ev_*\colon \pi_1(\mathcal{E}(M)) \to \pi_1(M, x_0))$ has been studied by Gottlieb [7]. It is always a central subgroup of $\pi_1(M, x_0)$, and $G(M, x_0)$ is trivial if $\chi(M) \neq 0$ (e.g. when $\pi_1(M, x_0)$ is finite).

Since M is connected, we see that any homotopy equivalence is homotopic to a base-point preserving homotopy equivalence. We are studying $\mathrm{Aut}_\bullet(M) := \pi_0(\mathcal{E}_\bullet(M))$. Notice that the composition

$$\pi_1(M, x_0) \to \pi_0(\mathcal{E}_\bullet(M)) \to \mathrm{Aut}(\pi_1(M, x_0))$$

just sends an element $\sigma \in \pi_1(M, x_0)$ to the automorphism "conjugation by σ".

The inclusion gives a fixed reference map $c\colon M \to B$ which is base-point preserving, and induces a homomorphism $\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B)$, by obstruction theory. We also have a map

$$\alpha\colon \ \mathrm{Aut}_\bullet(M) \to \Omega_4^{Spin}(M)$$

defined by $\alpha(f) := [M, f] - [M, id]$, but this is *not* a homomorphism:

$$\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g)) . \tag{2.1}$$

Since f is orientation-preserving, the fundamental class $f_*[M] = [M]$ and so the image of α is contained in the modified bordism group $\widehat{\Omega}_4^{Spin}(M)$.

We have already mentioned the modified relative bordism groups $\widehat{\Omega}_5^{Spin}(B, M)$, where the representing objects (W, F) are spin manifolds of dimension 5 with boundary, such that $f = F|\partial W$ has degree zero. The usual bordism exact sequence of the pair (B, M) can be adapted to include these modified groups.

Lemma 2.2. *There is an exact sequence*

$$\ldots \Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B) \to \widehat{\Omega}_5^{Spin}(B, M) \to \widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B) .$$

Proof. Left to the reader. □

2.2 The groups $\widetilde{\mathcal{H}}(M)$

Next we define the groups $\widetilde{\mathcal{H}}(M)$ as the bordism groups of objects (W, F) where W is a compact 5-dimensional spin manifold with $\partial_1 W = -M$ and $\partial_2 W = M$, and $F: W \to M$ is a continuous map such that $F|\partial_1 W = id_M$ and $F|\partial_2 W = f$ is a base-point and orientation-preserving homotopy equivalence. In particular, we mean that the spin structure on W is a lift of ν_W to $BSpin$ which agrees with our fixed lift for ν_M on both boundary components $\partial_1 W$ and $\partial_2 W$. We do not, however, require the self-equivalence f to preserve the spin structure on M. Two such objects (W, F) and (W', F') are *bordant* if there is a base-point preserving homotopy h between $f = F|\partial_2 W$ and $f' = F'|\partial_2 W'$, such that the closed, spin 5-manifold

$$(-W' \cup_{\partial_1 W' = \partial_1 W} W \cup_{\partial_2 W = M \times 0 \amalg \partial_2 W' = M \times 1} M \times I, F' \cup F \cup h) \tag{2.3}$$

represents zero in $\Omega_5^{Spin}(M)$. We define a group structure on $\widetilde{\mathcal{H}}(M)$ by the formula

$$(W, F) \bullet (W', F') := (W \cup_{\partial_2 W = \partial_1 W'} W', F \cup f \circ F') . \tag{2.4}$$

This is easily seen to be well-defined, and the inverse of (W, F) is represented by $(-W, f^{-1} \circ F)$ where f^{-1} is a base-point preserving homotopy inverse for $f = F|\partial_2 W$. By convention, $\partial_1(-W) = \partial_2(W)$, so to obtain an object of the form required we must adjoin a collar $M \times I$ to $-W$ along $\partial_1(-W)$ mapped into M by a homotopy between $f^{-1} \circ f$ and id_M. The different choices of such a homotopy result in bordant representatives for the inverse. The identity element in this group structure is represented by the bordism $(M \times I, p_1)$, where $p_1: M \times I \to M$ is the projection on the first factor. There is a homomorphism $\Omega_5^{Spin}(M) \to \widetilde{\mathcal{H}}(M)$ by taking the disjoint union of a closed, spin 5-manifold mapping into M and the identity element $(M \times I, p_1)$, and a homomorphism $\widetilde{\mathcal{H}}(M) \to \mathrm{Aut}_\bullet(M)$ mapping (W, F) to the homotopy class of $f := F|\partial_2 W$.

Lemma 2.5. *There is an exact sequence of pointed sets*

$$\Omega_5^{Spin}(M) \longrightarrow \widetilde{\mathcal{H}}(M) \longrightarrow \mathrm{Aut}_\bullet(M) \xrightarrow{\alpha} \widehat{\Omega}_4^{Spin}(M)$$

where only the last map α fails to be a group homomorphism.

Proof. Left to the reader. □

2.3 The groups $\widetilde{\mathcal{H}}(B)$

To obtain a similar exact sequence through $\mathrm{Aut}_\bullet(B)$ we start by defining the group $\widetilde{\mathcal{H}}(B)$ as the bordism group of objects (W, F) where W is a compact 5-dimensional spin manifold with $\partial_1 W = -M$ and $\partial_2 W = M$, and $F: W \to B$ is a continuous map such that $F|\partial_1 W = c$ and $F|\partial_2 W = f$ is a base-point preserving 3-equivalence. Two such objects (W, F) and (W', F') are *bordant* if there is a base-point preserving homotopy h into B between $f = F|\partial_2 W$ and $f' = F'|\partial_2 W'$, such that the closed, spin 5-manifold (2.3) represents zero in $\Omega_5^{Spin}(B)$. To define the group structure on $\widetilde{\mathcal{H}}(B)$ we first remark that a base-point preserving 3-equivalence $f\colon M \to B$ induces a base-point preserving self-equivalence $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. Furthermore, the map ϕ_f is uniquely defined by this equation, up to a base-point preserving homotopy.

The multiplication is now defined as in (2.4) by the formula

$$(W, F) \bullet (W', F') := (W \cup_{\partial_2 W = \partial_1 W'} W', F \cup \phi_f \circ F')$$

and the identity element is represented by $(M \times I, c \circ p_1)$. The inverse of (W, F) is represented by $(-W, \phi_f^{-1} \circ F)$ where ϕ_f^{-1} is a base-point preserving homotopy inverse for the self-equivalence $\phi_f\colon B \to B$ induced by $f = F|\partial_2 W$.

Lemma 2.6. $\widetilde{\mathcal{H}}(M) \cong \widetilde{\mathcal{H}}(B)$.

Proof. We have a well-defined homomorphism $\widetilde{\mathcal{H}}(M) \to \widetilde{\mathcal{H}}(B)$ by composing with our reference map $c\colon M \to B$. Suppose that (W, F) represents an element in $\widetilde{\mathcal{H}}(B)$. Since $F|\partial_1 W = c$, the identity map on M is a lift of $F|\partial_1 W = c$ over M. We want to extend this lift over W by homotoping the map $F\colon W \to B$ into M, relative to $\partial_1 W$. By low-dimensional surgery on the map F, we may assume that F is 2-connected. If X denotes the homotopy fibre of c, we are looking for the obstructions to lifting the map F relative to $\partial_1 W$ to the total space of the fibration $X \to M \to B$. But the fibre X is 2-connected, so the lifting obstructions lie in $H^{i+1}(W, \partial_1 W; \pi_i(X))$ for $i \geq 3$. However, $H^{i+1}(W, \partial_1 W) = H_{5-i-1}(W; \partial_2 W) = 0$ if $i \geq 3$, for any coefficients, since F is 2-connected and f is 3-connected. If $\hat{F}\colon W \to M$ is a lift of F, then $\hat{f} = \hat{F}|\partial_2 M$ is a 3-equivalence, and has degree 1 since it is bordant over M to the identity map. Therefore $\hat{f}$ is an orientation and base-point preserving homotopy equivalence. This proves that the natural map $\widetilde{\mathcal{H}}(M) \to \widetilde{\mathcal{H}}(B)$ is *surjective*.

Suppose now that (W, F) and (W', F') represent two elements in $\widetilde{\mathcal{H}}(M)$ which are bordant over B. We may assume that the reference map $T \to B$ for the bordism is 3-connected (by surgery on the interior of T), and then it follows as above that there are no obstructions to lifting this reference map to M, relative to the union of the boundary components (W, F), (W', F') and $\partial_1 W \times I$. A lifting of the reference map $T \to B$ restricted to $\partial_2 W \times I$ gives a base-point preserving homotopy between f and f' as required. Therefore (W, F) and (W', F') are bordant over M, and the map $\widetilde{\mathcal{H}}(M) \to \widetilde{\mathcal{H}}(B)$ is *injective*. □

2.4 The map β

We have a map of pointed sets

$$\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)$$

defined by $\beta(\phi\colon B \to B) := [M, \phi \circ c] - [M, c]$. We also have a homomorphism

$$\pi_1(\mathcal{E}_\bullet(B)) \to \Omega_5^{Spin}(B)$$

sending the adjoint map $h\colon B \times S^1 \to B$, for a representative of an element in $\pi_1\mathcal{E}_\bullet(B)$, to the bordism element $[M \times S^1, h \circ (c \times id)]$. We use the null-bordant spin structure on the S^1 factor. To see that this map induces a group homomorphism, consider the surface F obtained from the 2-disk by removing two small open balls in the interior. If $h, h'\colon B \times S^1 \to B$ are the adjoints of maps representing elements of $\pi_1(\mathcal{E}_\bullet(B))$, and $h'' = h \bullet h'$ is the adjoint of the product, then there is an obvious map from $B \times F \to B$ such that the restriction to the boundary is given by h, h' on the boundaries of the interior balls and by h'' on the exterior boundary component. Then $M \times F$ gives the required spin bordism.

Finally, the homomorphism $\Omega_5^{Spin}(B) \to \widetilde{\mathcal{H}}(M)$ is defined taking the disjoint union of a closed, spin 5-manifold mapping into B and the identity element $(M \times I, c \circ p_1)$.

Lemma 2.7. *There is an exact sequence of pointed sets*

$$\pi_1(\mathcal{E}_\bullet(B)) \longrightarrow \Omega_5^{Spin}(B) \longrightarrow \widetilde{\mathcal{H}}(M) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{\ \beta\ } \Omega_4^{Spin}(B)$$

where only the last map β fails to be a group homomorphism.

Proof. We will prove exactness for the related sequence where $\widetilde{\mathcal{H}}(M)$ is replaced by $\widetilde{\mathcal{H}}(B)$, and then apply Lemma 2.6. It follows easily from the definitions that the composite of any two maps in this new sequence is trivial, and that we have exactness at the terms $\widetilde{\mathcal{H}}(B)$ and $\mathrm{Aut}_\bullet(B)$. It remains to check exactness at $\Omega_5^{Spin}(B)$. Let (N, g) represent an element of $\Omega_5^{Spin}(B)$ which maps to the identity in $\widetilde{\mathcal{H}}(B)$. This means that the bordism element $(N, g) \amalg (M \times I, c \circ p_1)$ is bordant to $(M \times I, c \circ p_1)$. In particular, there is a base-point preserving homotopy $h\colon M \times I \to B$ with $h|(M \times 0) = c$ and $h|(M \times 1) = c$. This homotopy h induces a pointed homotopy $\hat{h}\colon B \times I \to B$ from the identity to the identity, representing an element of $\pi_1(\mathcal{E}_\bullet(B))$. □

2.5 The map γ

The remaining exact sequence in our braid diagram involves the construction of a map $\gamma\colon \widehat{\Omega}_5^{Spin}(B, M) \to \mathrm{Aut}_\bullet(M)$. Let (W, F) denote an element of $\widehat{\Omega}_5^{Spin}(B, M)$. This is a 5-dimensional spin manifold with boundary $(W, \partial W)$, equipped with a reference map $F\colon W \to B$ such that $F|\partial W$ factors through the classifying map $c\colon M \to B$. We may assume that ∂W is connected.

By taking the boundary connected sum with the zero bordant element $(M \times I, p_1)$ along ∂W and $M \times 1$, we may assume that W has two boundary components $\partial_1 W = -M$ and $\partial_2 W = N$ with the reference map $F|\partial_1 W = c$. We may assume (by low-dimensional surgery on the map F) that F is a 2-equivalence. Moreover, since (W, F) is a modified bordism element, $g := F|\partial_2 W$ is a degree 1 map from $N \to M$.

Now consider the obstructions to lifting the map $F\colon W \to B$ to M, relative to $F|\partial_2 W$. These obstructions lie in the groups $H^{i+1}(W, N; \pi_i(X))$, where X denotes the fibre of the map $c\colon M \to B$. Since X is 2-connected, after applying Poincaré duality we see that the lifting obstructions lie in the groups $H_{5-i-1}(W, M; \pi_i(X)) = 0$, for $i \geq 3$. Let $r\colon W \to M$ be a lift of F relative to N, and consider the map $f := r|\partial_1 W\colon M \to M$. Since $c \circ r \simeq F$, and $F|\partial_1 W = c$, we have $c \circ f \simeq c$ and hence f is a 3-equivalence. However $f_*[M] = g_*[N] = [M]$ since g has degree 1 and the maps f and g are bordant over M. However, a degree 1 map $f\colon M \to M$ which is a 3-equivalence is a homotopy equivalence (by Poincaré duality and Whitehead's Theorem). We may assume that f is also base-point preserving

Lemma 2.8. *There is a well-defined map*

$$\gamma\colon \widehat{\Omega}_5^{Spin}(B, M) \to \mathrm{Aut}_\bullet(M)\ .$$

Proof. Let $r\colon W \to M$ be a lifting of F and let $f := r|\partial_1 W$. We define

$$\gamma(W, F) := [f\colon M \to M] \in \mathrm{Aut}_\bullet(M)\ .$$

To see that the map γ is well-defined, suppose that (W', F') is another representative for the same relative bordism class and that we have already found liftings r and r' of the maps F and F' respectively. Let (T, φ) denote a bordism between (W, F) and (W', F'), respecting the boundary. More precisely, ∂T consists of the union of W, W', $M \times I$ and a 5-dimensional bordism P between N and N'. We may assume that the reference map $\varphi\colon T \to B$ is a 3-equivalence by surgery on the interior of T. Now consider the obstructions to lifting the map φ to M, relative both to $\varphi|P$ and to our chosen liftings r and r'. The obstructions to lifting φ lie in the groups $H^{i+1}(T, W \cup P \cup W'; \pi_i(X))$ for $i \geq 3$. They may be evaluated by Poincaré duality as above, and are again zero. Then any such lifting $\hat{\varphi}\colon T \to M$ of φ gives a homotopy $h = \hat{\varphi}|(M \times I)$ between $f = r|\partial_1 W$ and $f' = r'|\partial_1 W'$. We may assume in addition that h preserves the base-point x_0, by constructing our lifting relative to a thickening $D^4 \times I \subset M \times I$ of the interval $x_0 \times I$. □

To check that the map γ fits into our braid diagram, we introduce another object. Let $\mathcal{H}(B)$ denote the equivalence classes of triples $(M \times I, h, f)$, where $f\colon M \to M$ is a base-point preserving homotopy equivalence, and $h\colon M \times I \to B$ is a base-point preserving homotopy between c and $c \circ f$. Two triples $(M \times I, h, f)$ and $(M \times I, h', f')$ are equivalent if there is a base-point preserving homotopy $p\colon M \times I \to M$ between f and f', and a continuous map $t\colon M \times I \times I \to B$ such that $t|M \times I \times 0 = h$, $t|M \times I \times 1 = h'$, $t|M \times 0 \times I = c$ and $t|M \times 1 \times I = c \circ p$. We define a multiplication on $\mathcal{H}(B)$ by the union

$$(M \times I, h, f) \bullet (M \times I, h', f') = (M \times I, h \cup h' \circ f, f' \circ f)$$

where the two copies of $M \times I$ on the left-hand side are identified with $M \times [0, 1/2]$ and $M \times [1/2, 1]$ respectively on the right-hand side. The inverse of $(M \times I, h, f)$ is represented by $(M \times I, \bar{h} \circ f^{-1}, f^{-1})$, where $\bar{h}(x, t) = h(x, 1-t)$, for $0 \leq t \leq 1$ and $x \in M$, and f^{-1} is a base-point preserving homotopy inverse for f. We adjoin a pointed homotopy between $f \circ f^{-1}$ at the end $M \times 0$ to obtain an element in the standard form.

Lemma 2.9. *There is an exact sequence of groups and homomorphisms*

$$\pi_1(\mathcal{E}_\bullet(B)) \longrightarrow \mathcal{H}(B) \xrightarrow{\partial} \mathrm{Aut}_\bullet(M) \longrightarrow \mathrm{Aut}_\bullet(B) \ .$$

Proof. It is clear that the map $(M \times I, h, f) \mapsto f$ gives a homomorphism $\partial\colon \mathcal{H}(B) \to \mathrm{Aut}_\bullet(M)$, and the exactness is just a formal consequence of the definitions. In particular, $\mathrm{Im}\,\partial$ is a normal subgroup of $\mathrm{Aut}_\bullet(M)$. □

Remark 2.10. For each $[g] \in \mathrm{Aut}_\bullet(M)$, we have a base-point preserving self-equivalence $\phi_g\colon B \to B$ such that $c \circ g = \phi_g \circ c$. There is a conjugation action on $\mathcal{H}(B)$ defined by

$$(M \times I, h, f) \mapsto (M \times I, \phi_g \circ h \circ g^{-1}, g \circ f \circ g^{-1})$$

which is compatible with the boundary map $\partial\colon \mathcal{H}(B) \to \mathrm{Aut}_\bullet(M)$.

Lemma 2.11. *There is a bijection* $\eta\colon \widehat{\Omega}_5^{Spin}(B, M) \cong \mathcal{H}(B)$ *such that* $\partial \circ \eta = \gamma$.

Proof. Let (W, F) represent an element of $\widehat{\Omega}_5^{Spin}(B, M)$, as constructed at the beginning of this sub-section, with F a 2-connected map as usual. We have constructed a lifting $r\colon W \to M$ of F relative to $g := F|\partial_2 W$. In other words, F is homotopic to r over B and we can use a homotopy to give a map $\varphi\colon W \times I \to B$ such that $\varphi|W \times 0 = F$, $\varphi|W \times 1 = r$, and $\varphi|\partial_2 \times I = g$. Let $h := \varphi|\partial_1 W \times I$. Then $h\colon M \times I$ is a homotopy between c and $c \circ f$ where $f := r|\partial_1 W$. We define a map $\eta\colon \widehat{\Omega}_5^{Spin}(B, M) \to \mathcal{H}(B)$ by $(W, F) \mapsto (M \times I, h, f)$. It is easy to check that this map is well-defined and gives a bijection between the two sets. By construction, the map $\gamma = \partial \circ \eta$ is the composite of this bijection and the boundary map $\partial\colon \mathcal{H}(B) \to \mathrm{Aut}_\bullet(M)$ from Lemma 2.9. □

Remark 2.12. This argument shows that the element (W, F) is bordant to $(M \times I, h)$, so it represents the same bordism class in $\widehat{\Omega}_5^{Spin}(B, M)$. However, we do not know if the bordism group structure (addition by disjoint union) agrees with the multiplication $\bullet$ defined on the elements $(M \times I, h)$. For this reason, we don't know if the map γ is always a homomorphism. If $\pi_1(M, x_0)$ has odd order, it turns out that γ *is* a homomorphism.

Corollary 2.13. *There is an exact sequence of pointed sets*

$$\pi_1(\mathcal{E}_\bullet(B)) \longrightarrow \widehat{\Omega}_5^{Spin}(B, M) \xrightarrow{\gamma} \mathrm{Aut}_\bullet(M) \longrightarrow \mathrm{Aut}_\bullet(B) \ .$$

Proof. Left to the reader. □

2.6 Commutativity of the braid

We have now verified the exactness of all the sequences in the braid diagram, so it remains to check the commutativity of the diagram. We will only discuss two of the sub-diagrams.

Lemma 2.14. *The composite $\alpha \circ \gamma$ equals the boundary map $\partial \colon \widehat{\Omega}_5^{Spin}(B, M) \to \widehat{\Omega}_4^{Spin}(M)$.*

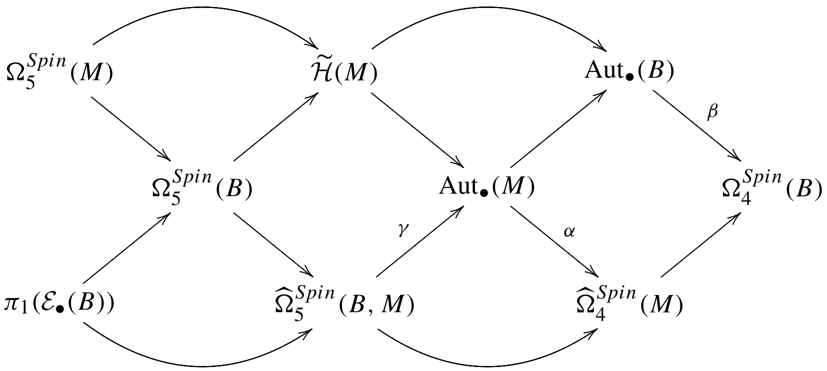
Proof. Let (W, F) represent an element of $\widehat{\Omega}_5^{Spin}(B, M)$ in the standard form above. Then its image in $\widehat{\Omega}_4^{Spin}(M)$ is represented by $[N, g] - [M, id]$, where $g := F|\partial_2 W$ as usual. However, the existence of a lifting $r\colon W \to M$ for F shows that $[N, g]$ is bordant over M to $[M, f]$, and so $\partial(W, F)$ represents the same bordism element as $\alpha \circ \gamma(W, F)$. □

Lemma 2.15. *The composite $\Omega_5^{Spin}(B) \to \widetilde{\mathcal{H}}(M) \to$ $\mathrm{Aut}_\bullet(M)$ equals the composite $\Omega_5^{Spin}(B) \to \widehat{\Omega}_5^{Spin}(B, M) \to$ $\mathrm{Aut}_\bullet(M)$ up to inversion $[f] \mapsto [f]^{-1}$ in* $\mathrm{Aut}_\bullet(M)$.

Proof. Let (N, g) denote an element of $\Omega_5^{Spin}(B)$. Then we map it into $\widetilde{\mathcal{H}}(M)$ by forming the connected sum $(M \times I \sharp N, p_1 \sharp g)$ and lifting the map $\varphi = p_1 \sharp g$ to $\hat{\varphi}\colon M \times I \sharp N \to M$ relative to $M \times 0$. Then $[\hat{\varphi}|M \times 1]$ is the image of the first composition in $\mathrm{Aut}_\bullet(M)$. To compute the other composition, we again form the connected sum $(M \times I \sharp N, p_1 \sharp g)$ and lift the map $\varphi = p_1 \sharp g$ to $r\colon M \times I \sharp N \to M$ relative to $M \times 1$. Then the image of the second composition is represented by $f := r|M \times 0$. However, notice that the map $f^{-1} \circ r$, together with a pointed homotopy from $f^{-1} \circ f$ in a small collar of $M \times 0$, gives another lifting of φ relative to $M \times 0$. Therefore $[\hat{\varphi}|M \times 1] = [f^{-1}]$, showing that the two compositions agree up to inversion in $\mathrm{Aut}_\bullet(M)$. □

We have proved that our braid diagram is *sign-commutative*, meaning that the sub-diagrams are all strictly commutative except for the two composites ending in $\mathrm{Aut}_\bullet(M)$ which only agree up to inversion.

Theorem 2.16. *Let M be a closed, oriented smooth (respectively topological)* 4-*manifold. If M is a spin manifold, there is a sign-commutative diagram of exact sequences*



involving the bordism groups of smooth (respectively topological) spin manifolds. All the maps except α, β and possibly γ are group homomorphisms.

3 Non-spin bordism groups

When $w_2(M) \neq 0$ the bordism groups must be modified in the above braid in order to carry out the arguments used to establish commutativity.

Let $\xi : E \to BSO$ be a fibration, and recall that elements in the bordism groups $\Omega_n(E)$ are represented by maps $\bar{\nu} \colon N \to E$ from a smooth, closed, n-manifold N into E, such that $\xi \circ \bar{\nu} = \nu_N$, where $\nu_N : N \to BSO$ classifies the stable normal bundle of N. The bordism relation also involves a compatible lifting of the normal bundle data over the cobordism (see [15], [18, Chap. II]).

Recall that a *normal k-smoothing* of M in E is a lifting $\bar{\nu} \colon M \to E$ of ν_M such that $\bar{\nu}$ is a $(k+1)$-equivalence [12, p. 711]. The fibration $E \to BSO$ is called k-universal if its fibre is connected, with homotopy groups vanishing in dimensions $\geq k+1$. The *normal* 2*-type* of M is a 2-universal fibration $E \to BSO$ admitting a normal 2-smoothing of M (see [12, p. 711] for an extensive development of these concepts).

For the non-spin case of our braid we will use the bordism groups of the normal 2-type:

$$\begin{array}{ccccc} BSpin & \xrightarrow{i} & B\langle w_2\rangle & \xrightarrow{j} & B \\ \| & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle w_2} \\ BSpin & \longrightarrow & BSO & \xrightarrow{w} & K(\mathbf{Z}/2, 2) \end{array}$$

as described in [20, §2]. The map $w = w_2(\gamma)$ pulls back the second Stiefel-Whitney class for the universal oriented vector bundle γ over BSO. The "James" spectral sequence used to compute $\Omega_*(B\langle w_2\rangle) = \pi_*(M\xi)$ has the same E_2-term as the one used above for $w_2 = 0$, but the differentials are twisted by w_2. In particular, d_2 is the dual of Sq^2_w, where $Sq^2_w(x) := Sq^2(x) + x \cup w_2$. There is a corresponding non-spin version of $\Omega_*^{Spin}(M)$, namely the bordism groups $\Omega_*(M\langle w_2\rangle) := \pi_*(M\xi)$ of the Thom space associated to the fibration:

$$\begin{array}{ccccc} BSpin & \xrightarrow{i} & M\langle w_2\rangle & \xrightarrow{j} & M \\ \| & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle w_2} \\ BSpin & \longrightarrow & BSO & \xrightarrow{w} & K(\mathbf{Z}/2, 2) \end{array}$$

Again the E_2-term of the James spectral sequence is unchanged from the spin case, but the differentials are twisted by w_2 with the above formula for Sq^2_w. As in the spin case, we choose a particular representative for the map w_2 such that $w_2 = w \circ \nu_M$.

Our next step is to define a suitable "thickening" of $\mathrm{Aut}_\bullet(M)$ for the non-spin case. Here is the main technical ingredient.

Lemma 3.1. *Let $f \colon M \to M$ be a base-point and orientation-preserving homotopy equivalence. Then there exists a base-point preserving homotopy equivalence $f' \colon M \to M$, such that $f \simeq f'$ preserving the base point, with $w \circ \nu_M = w_2 \circ f'$.*

Proof. By the Dold-Whitney Theorem [5], there is an isomorphism $f^*(\nu_M) \cong \nu_M$. We therefore have a (base-point preserving) homotopy $h\colon M \times I \to BSO$ between the classifying maps $\nu_M \circ f \simeq \nu_M$. Now define $\hat{f}\colon M \to M\langle w_2\rangle$ lifting $\nu_M \circ f$ by the formula

$$\hat{f}(x) := (f(x), \nu_M(f(x))$$

for all $x \in M$, and note that this makes sense because $w_2 = w \circ \nu_M$ as maps to $K(\mathbf{Z}/2, 2)$. We apply the covering homotopy theorem to get $\hat{h}\colon M \times I \to M\langle w_2\rangle$ lifting h, with the property that $\xi \circ (\hat{h}\,|\,_{M\times 1}) = \nu_M$. Let $f'\colon M \to M$ be defined by the formula $f' := j \circ (h\,|\,_{M\times 1})$, where $j\colon M\langle w_2\rangle \to M$ is the projection on the first factor. Then $f' \simeq f$ by the homotopy $j \circ \hat{h}$, and we have

$$w_2(f'(x)) = w_2(j(\hat{h}\,|\,_{M\times 1}(x))) = w(\xi(\hat{h}\,|\,_{M\times 1}(x))) = w(\nu_M(x))$$

for all $x \in M$, as required. □

As a consequence of the Lemma, the formula $\hat{f}'(x) := (f'(x), \nu_M(x))$ gives a map $\hat{f}'\colon M \to M\langle w_2\rangle$, and in fact $\hat{f}' \equiv \hat{h}\,|\,_{M\times 1}$ by construction. Therefore $\xi \circ \hat{f}' = \nu_M$ as maps $M \to BSO$. We will consider the set of all such maps into $M\langle w_2\rangle$ under a suitable equivalence relation.

Definition 3.2. *Let* $\mathrm{Aut}_\bullet(M, w_2)$ *denote the set of equivalence classes of maps* $\hat{f}\colon M \to M\langle w_2\rangle$ *such that (i)* $f := j \circ \hat{f}$ *is a base-point and orientation preserving homotopy equivalence, and (ii)* $\xi \circ \hat{f} = \nu_M$. *Two such maps* $\hat{f}$ *and* $\hat{g}$ *are* equivalent *if there exists a homotopy* $\hat{h}\colon M \times I \to M\langle w_2\rangle$ *such that* $h := j \circ \hat{h}$ *is a base-point preserving homotopy between* f *and* g*, and* $\xi \circ \hat{h} = \nu_M \circ p_1$*, where* $p_1\colon M \times I \to M$ *denotes projection on the first factor.*

Given two maps $\hat{f}, \hat{g}\colon M \to M\langle w_2\rangle$ as above, we define

$$\hat{f} \bullet \hat{g}\colon M \to M\langle w_2\rangle$$

as the unique map from M into the pull-back $M\langle w_2\rangle$ defined by the pair $f \circ g\colon M \to M$ and $\nu_M\colon M \to BSO$. Since $w_2 \circ f \circ g = w \circ \nu_M \circ g = w_2 \circ g = w \circ \nu_M$, this pair of maps is compatible with the pull-back.

Lemma 3.3. $\mathrm{Aut}_\bullet(M, w_2)$ *is a group under this operation.*

Proof. To check that the operation just defined passes to equivalence classes, suppose that $\hat{h}$ is a homotopy as above between $\hat{f}$ and $\hat{f}'$ representing the same element of $\mathrm{Aut}_\bullet(M, w_2)$. Let $h := j \circ \hat{h}$ and notice that $w_2 \circ h \circ g = w \circ \nu_M \circ p_1 \circ (g \times id) = w_2 \circ g \circ p_1 = w \circ \nu_M \circ p_1$. We have a similar argument in the case when $\hat{g}$ is varied by a homotopy.

Next we discuss the identity element and inverses. Let $\widehat{id}_M\colon M \to M\langle w_2\rangle$ denote the map defined by the pair $(id_M\colon M \to M, \nu_M\colon M \to BSO)$. This map will represent the identity element in our group structure.

Given $\hat{f}$ representing an element of $\mathrm{Aut}_\bullet(M, w_2)$, let $\hat{g}\colon M \to M\langle w_2\rangle$ be a map constructed as in Lemma 3.1 applied to any base-point preserving homotopy inverse f^{-1} for $f := j \circ \hat{f}$. Now if $h\colon M \times I \to M$ is a base-point preserving homotopy between $f \circ g$ and id_M, we can assume that $w_2 \circ h = w_2 \circ p_1$. To see this, note that the different maps $M \times I \to K(\mathbf{Z}/2, 2)$ relative to the given maps on the boundary are classified by $H^1(M; \mathbf{Z}/2)$. But we can construct a map $M \times S^1 \to M$ using any element of $\pi_1(M, x_0)$, and this gives a homotopy from id_M to itself realizing any desired element of $H^1(M; \mathbf{Z}/2)$. It follows that the pair of maps $h\colon M \times I \to M$ and $\nu_M \circ p_1\colon M \times I \to BSO$ define a unique map $\hat{h}\colon M \times I \to M\langle w_2\rangle$. This is exactly the required homotopy between $\hat{f} \circ \hat{g}$ and $\widehat{id}_M$. We will refer to $\hat{h}$ as an *admissible* homotopy. Checking the remaining properties of the group structure will be left to the reader. □

Now we will define a map

$$\alpha\colon \operatorname{Aut}_\bullet(M, w_2) \to \widehat{\Omega}_4(M\langle w_2\rangle)$$

for use in our braid, where the modified bordism groups are defined by letting the *degree* of a reference map $\hat{g}\colon N^4 \to M\langle w_2\rangle$ be the ordinary degree of $g := j \circ \hat{g}$. Given $[\hat{f}] \in \operatorname{Aut}_\bullet(M, w_2)$, let

$$\alpha(\hat{f}) := [M, \hat{f}] - [M, \widehat{id}_M] \in \Omega_4(M\langle w_2\rangle),$$

and notice that this element has degree zero. Since $\xi \circ \hat{f} = \nu_M$, we have a bundle map $\hat{b}\colon \nu_M \to \xi$ and a commutative diagram

$$\begin{array}{ccc} E(\nu_M) & \xrightarrow{\hat{b}} & E(\xi) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\hat{f}} & M\langle w_2\rangle \end{array}$$

expressing that fact that $(M, \hat{f})$ represents an element of the bordism theory for the normal 2-type. It is clear from the way that the equivalence relation is defined for $\operatorname{Aut}_\bullet(M, w_2)$ that α is well-defined, independent of the choice of representative for $[\hat{f}]$.

Next comes the definition of $\widetilde{\mathcal{H}}(M, w_2)$ and the homomorphism $\widetilde{\mathcal{H}}(M, w_2) \to \operatorname{Aut}_\bullet(M, w_2)$.

Definition 3.4. *Let $\widetilde{\mathcal{H}}(M, w_2)$ denote the bordism groups of pairs $(W, \widehat{F})$, where W is a compact, oriented 5-manifold with $\partial_1 W = -M$ and $\partial_2 W = M$. The map $\widehat{F}\colon W \to M\langle w_2\rangle$ restricts to $\widehat{id}_M$ on $\partial_1 W$, and on $\partial_2 W$ to a map $\hat{f}\colon M \to M\langle w_2\rangle$ satisfying properties (i) and (ii) of Definition 3.2.*

Two such objects $(W, \widehat{F})$ and $(W', \widehat{F}')$ are *bordant* if there is an equivalence $\hat{h}$ between $\hat{f} = \widehat{F}|\partial_2 W$ and $\hat{f}' = \widehat{F}'|\partial_2 W'$, such that the closed 5-manifold

$$(-W' \cup_{\partial_1 W' = \partial_1 W} W \cup_{\partial_2 W = M \times 0 \amalg \partial_2 W' = M \times 1} M \times I, \widehat{F}' \cup \widehat{F} \cup h) \qquad (3.5)$$

represents zero in $\Omega_5(M\langle w_2\rangle)$. We define a group structure on $\widetilde{\mathcal{H}}(M, w_2)$ by the formula

$$(W, \widehat{F}) \bullet (W', \widehat{F}') := (W \cup_{\partial_2 W = \partial_1 W'} W', \widehat{F} \cup \hat{f} \bullet \widehat{F}')\,. \qquad (3.6)$$

This is easily seen to be well-defined, and the inverse of $(W, \widehat{F})$ is represented by $(-W, \hat{f}^{-1} \bullet \widehat{F})$ where $\hat{f}^{-1}$ represents the inverse for $\hat{f} = \widehat{F}|\partial_2 W$ in $\mathrm{Aut}_\bullet(M, w_2)$. By convention, $\partial_1(-W) = \partial_2(W)$, so to obtain an object of the form required we must adjoin a collar $M \times I$ to $-W$ along $\partial_1(-W)$ mapped into M by an admissible homotopy between $\hat{f}^{-1} \bullet \hat{f}$ and $\widehat{id}_M$. The different choices of such a homotopy result in bordant representatives for the inverse. The identity element in this group structure is represented by the bordism $(M \times I, \hat{p}_1)$, where $\hat{p}_1 := \widehat{id}_M \circ p_1$ and $p_1 \colon M \times I \to M$ is the projection on the first factor. There is a homomorphism $\Omega_5(M\langle w_2\rangle) \to \widetilde{\mathcal{H}}(M, w_2)$ by taking the disjoint union of a closed, 5-manifold with normal structure in $M\langle w_2\rangle$ and the identity element $(M \times I, \hat{p}_1)$.

Lemma 3.7. *There is an exact sequence of pointed sets*

$$\Omega_5(M\langle w_2\rangle) \longrightarrow \widetilde{\mathcal{H}}(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \xrightarrow{\alpha} \widehat{\Omega}_4(M\langle w_2\rangle)$$

where only the last map α fails to be a group homomorphism.

Proof. The homomorphism $\widetilde{\mathcal{H}}(M, w_2) \to \mathrm{Aut}_\bullet(M, w_2)$ is defined on representatives by sending $(W, \widehat{F})$ to $\hat{f} := \widehat{F}|\partial_2 W$. The rest of the details will be left to the reader. □

Finally, we will define the analogous bordism groups $\widetilde{\mathcal{H}}(B, w_2)$ and the group $\mathrm{Aut}_\bullet(B, w_2)$ of self-equivalences, together with the map $\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2\rangle)$. Here is the basic technical ingredient.

Lemma 3.8. *Given a base-point preserving map $f\colon M \to B$, there is a unique extension (up to base-point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. If f is a 3-equivalence then ϕ_f is a homotopy equivalence. If $w_2 \circ f = w_2$, then $w_2 \circ \phi_f = w_2$.*

Proof. The existence and uniqueness of the extension ϕ_f follow from obstruction theory, since $c\colon M \to B$ is a 3-equivalence. The other statements are clear. □

Definition 3.9. *Let* $\mathrm{Aut}_\bullet(B, w_2)$ *denote the set of equivalence classes of maps $\hat{f}\colon M \to B\langle w_2\rangle$ such that (i) $f := j \circ \hat{f}$ is a base-point preserving* 3*-equivalence, and (ii) $\xi \circ \hat{f} = \nu_M$. Two such maps $\hat{f}$ and $\hat{g}$ are* equivalent *if there exists a homotopy $\hat{h}\colon M \times I \to B\langle w_2\rangle$ such that $h := j \circ \hat{h}$ is a base-point preserving homotopy between f and g, and $\xi \circ \hat{h} = \nu_M \circ p_1$, where $p_1\colon M \times I \to M$ denotes projection on the first factor.*

Given two maps $\hat{f}, \hat{g}\colon M \to B\langle w_2\rangle$ as above, we define

$$\hat{f} \bullet \hat{g}\colon M \to B\langle w_2\rangle$$

as the unique map from M into the pull-back $B\langle w_2\rangle$ defined by the pair $\phi_f \circ \phi_g \circ c\colon M \to B$ and $\nu_M\colon M \to BSO$. Here we are using Lemma 3.8 to factor the maps $\phi_f \circ c = f$ and $\phi_g \circ c = g$. Since $w_2 \circ \phi_f \circ \phi_g \circ c = w \circ \nu_M \circ \phi_g \circ c = w_2 \circ \phi_g \circ c = w \circ \nu_M$, this pair of maps is compatible with the pull-back.

Lemma 3.10. $\mathrm{Aut}_\bullet(B, w_2)$ *is a group under this operation.*

Proof. Let $\hat{c}: M \to B\langle w_2 \rangle$ denote the map defined by the pair $(c: M \to B, \nu_M: M \to BSO)$. This map will represent the identity element in our group structure.

Given $\hat{f}$ representing an element of $\mathrm{Aut}_\bullet(B, w_2)$, write $f = \phi_f \circ c$ as above and choose a base-point preserving homotopy inverse $\psi: B \to B$ for ϕ_f, with the additional property that $w_2 \circ \psi = w_2$. This is another "lifting" argument using the fibration $B \to K(\mathbf{Z}/2, 2)$. Then the pair $g := \psi \circ c$ and ν_M define a map $\hat{g}: M \to B\langle w_2 \rangle$ representing the inverse of $\hat{f}$. We leave the check that $\hat{g} \bullet \hat{f} \simeq \hat{c}$ via an admissible equivalence to the reader. □

Definition 3.11. *Let $\widetilde{\mathcal{H}}(B, w_2)$ denote the bordism groups of pairs $(W, \widehat{F})$, where W is a compact, oriented 5-manifold with $\partial_1 W = -M$ and $\partial_2 W = M$. The map $\widehat{F}: W \to B\langle w_2 \rangle$ restricts to $\hat{c}$ on $\partial_1 W$, and on $\partial_2 W$ to a map $\hat{f}: M \to B\langle w_2 \rangle$ satisfying properties (i) and (ii) of Definition 3.9 .*

Two such objects $(W, \widehat{F})$ and $(W', \widehat{F}')$ are *bordant* if there is an equivalence $\hat{h}$ between $\hat{f} = \widehat{F}|\partial_2 W$ and $\hat{f}' = \widehat{F}'|\partial_2 W'$, such that the closed 5-manifold (3.5) represents zero in $\Omega_5(B\langle w_2 \rangle)$. We define a group structure on $\widetilde{\mathcal{H}}(B, w_2)$ as in (3.6) by the formula

$$(W, \widehat{F}) \bullet (W', \widehat{F}') := (W \cup_{\partial_2 W = \partial_1 W'} W', \widehat{F} \cup \hat{f} \bullet \widehat{F}') . \tag{3.12}$$

and the identity element is represented by $(M \times I, \hat{p}_1)$, where $\hat{p}_1 := \hat{c} \circ p_1$. The inverse of $(W, \widehat{F})$ is represented by $(-W, \hat{f}^{-1} \bullet \widehat{F})$ where $\hat{f}^{-1}$ represents the inverse for $\hat{f} = \widehat{F}|\partial_2 W$ in $\mathrm{Aut}_\bullet(B, w_2)$.

Lemma 3.13. $\widetilde{\mathcal{H}}(M, w_2) \cong \widetilde{\mathcal{H}}(B, w_2)$.

Proof. This follows as in the proof of Lemma 2.6 for the spin case: the lifting arguments take place over the fixed map $\nu_M: M \to BSO$. □

There is a homomorphism $\Omega_5(B\langle w_2 \rangle) \to \widetilde{\mathcal{H}}(B, w_2)$ by taking the disjoint union of a closed, 5-manifold with normal structure in $B\langle w_2 \rangle$ and the identity element $(M \times I, \hat{p}_1)$. Furthermore, we have a map $\beta: \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2 \rangle)$ defined by $\beta(\hat{f}) := [M, \hat{f}] - [M, \hat{c}]$.

We can also define $\mathcal{E}_\bullet(M, w_2)$ and $\mathcal{E}_\bullet(B, w_2)$ as the spaces of maps from $M \to M\langle w_2 \rangle$ or $M \to B\langle w_2 \rangle$ satisfying the properties (i) and (ii) of Definitions 3.2 or 3.9 respectively. Then $\mathrm{Aut}_\bullet(M, w_2) = \pi_0(\mathcal{E}_\bullet(M, w_2))$ and $\mathrm{Aut}_\bullet(B, w_2) = \pi_0(\mathcal{E}_\bullet(B, w_2))$. We therefore have a homomorphism $\pi_1(\mathcal{E}_\bullet(B, w_2)) \to \Omega_5(B\langle w_2 \rangle)$ sending the adjoint map $\hat{h}: M \times S^1 \to B\langle w_2 \rangle$ for a representative of an element in $\pi_1(\mathcal{E}_\bullet(B, w_2))$ to the bordism element $(M \times S^1, \hat{h})$ in the normal 2-type $B\langle w_2 \rangle$.

Lemma 3.14. *There is an exact sequence of pointed sets*

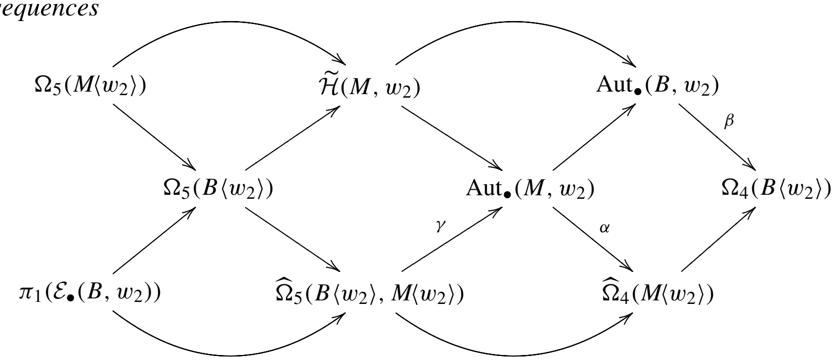
$$\pi_1(\mathcal{E}_\bullet(B, w_2)) \longrightarrow \Omega_5(B\langle w_2 \rangle) \longrightarrow \widetilde{\mathcal{H}}(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(B, w_2) \xrightarrow{\beta} \Omega_4(B\langle w_2 \rangle)$$

where only the last map β fails to be a group homomorphism.

Proof. Left to the reader. □

These definitions and properties allow is to establish our commutative braid.

Theorem 3.15. *Let M be a closed, oriented smooth (respectively topological)* 4-*manifold with normal* 2*-type $B\langle w_2\rangle$. There is a sign-commutative diagram of exact sequences*



involving the bordism groups of smooth (respectively topological) manifolds.

As before, the two composites ending in $\mathrm{Aut}_\bullet(M, w_2)$ agree up to inversion, and the other sub-diagrams are strictly commutative.

Proof. The proof of this result follows the pattern for the spin case. The key points are Lemma 3.13 and the definition of the map γ (see Lemma 2.8). □

We conclude this section by pointing out the connection between $\mathrm{Aut}_\bullet(M)$ and $\mathrm{Aut}_\bullet(M, w_2)$.

Lemma 3.16. *There is a short exact sequence of groups*

$$0 \to H^1(M; \mathbf{Z}/2) \to \mathrm{Aut}_\bullet(M, w_2) \to \mathrm{Aut}_\bullet(M) \to 1\ .$$

Proof. There is a natural map $\mathcal{E}_\bullet(M, w_2) \to \mathcal{E}_\bullet(M)$ defined by sending $\hat{f}$ to $f := j \circ \hat{f}$, and this induces a surjective homomorphism on the groups of homotopy classes over BSO. The identification of the kernel with $H^1(M; \mathbf{Z}/2)$ follows from the fibration $K(\mathbf{Z}/2, 1) \to M\langle w_2\rangle \to M \times BSO$ and obstruction theory. □

Remark 3.17. A similar result holds for $\mathrm{Aut}_\bullet(B, w_2)$, which maps surjectively onto the subgroup of $\mathrm{Aut}_\bullet(B)$ fixing w_2. The kernel is again isomorphic to $H^1(M; \mathbf{Z}/2)$.

4 Odd order fundamental groups

In this section, we assume that $\pi_1(M, x_0)$ is a finite group of odd order. We can then compute the terms in our braid to obtain a more explicit expression for $\mathrm{Aut}_\bullet(M) \cong \mathrm{Aut}_\bullet(M, w_2)$. We also have some information about the group $\mathcal{H}(M)$ of h-cobordisms.

Notice that (even without assumption on $\pi_1(M, x_0)$) there is an exact sequence

$$1 \to \mathcal{S}^h(M \times I, \partial) \to \mathcal{H}(M) \to \mathrm{Aut}_\bullet(M)$$

where $\mathcal{S}^h(M \times I, \partial)$ denotes the structure group of smooth or topological manifold structures on $M \times I$, relative to the given structure on $\partial(M \times I)$.

We first point out a useful input from surgery theory.

Lemma 4.1. *Suppose that $\pi_1(M)$ is finite of odd order. There is an injection $H_1(M;\mathbf{Z}) \to \widetilde{\mathcal{H}}(M, w_2)$, factoring through the map $\Omega_5(M\langle w_2\rangle) \to \widetilde{\mathcal{H}}(M, w_2)$ from the braid diagram.*

Proof. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & \tilde{L}_6(\mathbf{Z}[\pi_1]) & = & \tilde{L}_6(\mathbf{Z}[\pi_1]) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S}^h(M \times I, \partial) & \longrightarrow & \mathcal{H}(M) & \longrightarrow & \mathrm{Aut}_\bullet(M) \\
 & & \downarrow & & \downarrow & & \\
 & & H_1(M;\mathbf{Z}) & \longrightarrow & \widetilde{\mathcal{H}}(M, w_2) & & \\
 & & \downarrow & & \downarrow & & \\
 & & L_5(\mathbf{Z}[\pi_1]) & = & L_5(\mathbf{Z}[\pi_1]) & &
\end{array}$$

where the left-hand vertical sequence is from Wall's surgery exact sequence [22, Chap. 10]. To obtain the right-hand vertical sequence we use the modified surgery theory of [12]. The surgery obstruction map $\widetilde{\mathcal{H}}(M, w_2) \to L_5(\mathbf{Z}[\pi_1])$ from [12, Thm. 4] is the obstruction to finding a bordism over the normal type to an element of $\mathcal{H}(M)$ (see the remark on [12, p. 734] to replace the monoid $\ell_5(\mathbf{Z}[\pi_1])$ by the Wall group, and the remark on [12, p. 738] for the h-cobordism version). By construction, this map is a homomorphism. There is also an action of $L_6(\mathbf{Z}[\pi_1])$ on $\mathcal{H}(M)$, as in the surgery exact sequence, which again by construction gives a homomorphism. The exactness of the displayed right-hand sequence follows from [12, Thm. 3] and the remark [12, p. 730].

The horizontal maps come from the bordism interpretation of the surgery exact sequence

$$L_6(\mathbf{Z}[\pi_1]) \to \mathcal{S}^h(M \times I, \partial) \to \mathcal{T}(M \times I, \partial) \to L_5(\mathbf{Z}[\pi_1])$$

in which the normal invariant term $\mathcal{T}(M \times I, \partial)$ is the set of degree 1 normal maps $F\colon (W, \partial W) \to (M \times I, \partial)$, inducing the identity on the boundary [22, Prop. 10.2]. The group structure on this set is defined as for $\widetilde{\mathcal{H}}(M, w_2)$. The map $\mathcal{T}(M \times I, \partial) \to \widetilde{\mathcal{H}}(M, w_2)$ takes such an element to $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$. This map factors through $\Omega_5(M\langle w_2\rangle)$ by sending such an element to the bordism class of $(W \cup M \times I, \widehat{F})$. On the other hand, there is an isomorphism of groups

$$\mathcal{T}(M \times I, \partial) \cong [M \times I, \partial; G/TOP] = [SM; G/TOP]$$

when we use the *co-H*-space structure on the reduced suspension SM of M. That group structure agrees with the usual one for the normal invariants from the H-space structure on G/TOP (see [17, §1.6]).

A computation gives $[M \times I, \partial; G/TOP] \cong H_1(M;\mathbf{Z})$, and a diagram chase now shows that the composite map $H_1(M;\mathbf{Z}) \to \widetilde{\mathcal{H}}(M, w_2)$ is an injection. □

Remark 4.2. For later use, we will note that the map $H_1(M;\mathbf{Z}) \to \Omega_5(M\langle w_2\rangle)$ defined above may be identified with the homomorphism

$$H_1(M;\mathbf{Z}) = E_2^{1,4} \to E_\infty^{1,4} \subset \Omega_5(M\langle w_2\rangle)$$

in the Atiyah-Hirzebruch spectral sequence whose E_2-term is $H_p(M;\Omega_q^{Spin}(*))$.

To see this, we consider an embedding $f\colon (S^1\times D^3)\times I \to M\times I$ representing an element of $H_1(M;\mathbf{Z})$. There is a commutative diagram

$$\begin{array}{ccc}
\mathcal{T}(S^1\times D^4,\partial) & \longrightarrow & \Omega_5^{Spin}(S^1) \\
\downarrow & & \downarrow \\
\mathcal{T}(M\times I,\partial) & \longrightarrow & \Omega_5(M\langle w_2\rangle)
\end{array}$$

where the left vertical map is given by gluing a normal map with range $S^1\times D^4 \equiv S^1\times D^3\times I$ into $M\times I$, and extending by the identity. By Poincaré duality, there is a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{T}(S^1\times D^4,\partial) & \xleftarrow{\approx} & H^4(S^1\times D^4,\partial) & \xrightarrow{\approx} & H_1(S^1;\mathbf{Z}) & \longrightarrow & \Omega_5^{Spin}(S^1) \\
\downarrow & & \downarrow{\scriptstyle f^!} & & \downarrow{\scriptstyle f_*} & & \downarrow \\
\mathcal{T}(M\times I,\partial) & \xleftarrow{\approx} & H^4(M\times I,\partial) & \xrightarrow{\approx} & H_1(M;\mathbf{Z}) & \longrightarrow & \Omega_5(M\langle w_2\rangle)
\end{array}$$

factoring the one above, where $f^!$ denotes the map induced by the collapse $M\times I \to S^1\times D^4/S^1\times S^3$. The identification of $H_1(M;\mathbf{Z})$ with the normal invariants uses Poincaré duality with L-spectrum coefficients, but in this low dimensional situation it reduces to the ordinary duality. The last horizontal maps in this diagram are induced from the maps $E_2^{1,4} \to E_\infty^{1,4}$ in the spectral sequences.

The remaining proofs will be done in a number of steps, starting with the case of *spin* manifolds. We mean *topological* bordism throughout and homology with integral coefficients unless otherwise noted.

Proposition 4.3. *Let B denote the normal 2-type of a spin 4-manifold M with odd order fundamental group. Then $\Omega_4^{Spin}(B) \subset H_4(B)\oplus\mathbf{Z}$ and there is a short exact sequence $0 \to H_1(M) \to \Omega_5^{Spin}(B) \to H_5(B)$.*

Proof. This follows from the Atiyah-Hirzebruch spectral sequence, whose E_2-term is $H_p(B;\Omega_q^{Spin}(*))$. The first differential $d_2\colon E_2^{p,q} \to E_2^{p-2,q+1}$ is given by the dual of Sq^2 (if $q=1$) or this composed with reduction mod 2 (if $q=0$), see [20, p. 751]. We substitute the values $\Omega_q^{Spin}(*) = \mathbf{Z}, \mathbf{Z}/2, \mathbf{Z}/2, 0, \mathbf{Z}, 0$, for $0\le q\le 5$. Then the differential for $(p,q)=(4,1)$ becomes $d_2\colon H_4(B;\mathbf{Z}/2)\to H_2(B;\mathbf{Z}/2)$. This homomorphism may be detected by transfer to the universal covering $\tilde{B}$, since π_1 has odd order. Notice that $\tilde{B}$ is just a product of $\mathbb{C}P^\infty$'s. It follows that $Sq^2\colon H^2(\tilde{B};\mathbf{Z}/2)\to H^4(\tilde{B};\mathbf{Z}/2)$ is injective, hence its dual is surjective even when restricted to the subgroup of π_1-invariant elements (by averaging). Therefore, on the line $p+q=4$, the only groups which survive to E_∞ are $\mathbf{Z}$ in the $(0,4)$ position, and a subgroup of $H_4(B)$ in the $(4,0)$ position.

For the line $p+q=5$, we have again that the differential $d_2\colon H_6(B;\mathbf{Z}) \to H_4(B;\mathbf{Z}/2)$ from position $(p,q)=(6,0)$ is surjective onto the kernel of the above differential $d_2\colon H_4(B;\mathbf{Z}/2) \to H_2(B;\mathbf{Z}/2)$. This follows from the exactness of the sequence

$$H^2(\tilde{B};\mathbf{Z}/2) \xrightarrow{Sq^2} H^4(\tilde{B};\mathbf{Z}/2) \xrightarrow{Sq^2} H^6(\tilde{B};\mathbf{Z}/2)$$

and the surjectivity of $H_6(B;\mathbf{Z}) \to H_6(B;\mathbf{Z}/2)$. Finally, by transfer to $\tilde{B}$ we get $H_3(B;\mathbf{Z}/2)=0$. Therefore the groups that survive on this line are $H_1(B)=H_1(M)$ in the $(1,4)$ position (by Lemma 4.1) and $H_5(B)$ in the $(5,0)$ position. □

Lemma 4.4. $\ker(\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)) \subseteq \mathrm{Isom}([\pi_1,\pi_2,k,s])$.

Proof. Although β is not a homomorphism, we can still define $\ker(\beta)=\beta^{-1}(0)$. The natural map $\Omega_4^{Spin}(B) \to H_4(B)$ sends a bordism element to the image of its fundamental class. If $\phi \in \mathrm{Aut}_\bullet(B)$, and $c\colon M \to B$ is its classifying map, then $\beta(\phi) := [M,\phi\circ c] - [M,c]$. The image of this element in $H_4(B)$ is zero when $\phi_*(c_*[M]) = c_*[M]$. But $trf(c_*[M]) = s(M)$, the intersection form of M on π_2 considered as an element in $H_4(B)=\Gamma(\pi_2)$, so $\ker\beta$ is contained in the self-equivalences of B which preserve the quadratic 2-type. □

Next we calculate some more bordism groups and determine the image of the map $\alpha\colon \mathrm{Aut}_\bullet(M) \to \widehat{\Omega}_4^{Spin}(M)$.

Proposition 4.5. $\Omega_4^{Spin}(M) = \mathbf{Z}\oplus H_2(M;\mathbf{Z}/2)\oplus\mathbf{Z}$, *and* $\mathrm{Im}\,\alpha = H_2(M;\mathbf{Z}/2)$. $\Omega_5^{Spin}(M) = \mathbf{Z}/2\oplus H_1(M)$. *The map* $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B)$ *is projection onto the subgroup* $H_1(M)$.

Proof. We use the same spectral sequence, but the terms are a bit simpler because $H_3(M)=H_3(M;\mathbf{Z}/2)=H_5(M)=0$, and $H_4(M;\mathbf{Z}/2)=\mathbf{Z}/2$. Since M is spin, the map $Sq^2\colon H^2(M;\mathbf{Z}/2) \to H^4(M;\mathbf{Z}/2)$ is zero, so the differential $d_2\colon E_2^{4,1} \to E_2^{2,2}$ is also zero. The line $p+q=4$ now gives $\Omega_4^{Spin}(M) = \mathbf{Z}\oplus H_2(M;\mathbf{Z}/2)\oplus H_4(M)$. If $f\colon M\to M$ represents an element of $\mathrm{Aut}_\bullet(M)$, then $\alpha(f) := [M,f]-[M,id]$. It follows that $\alpha(f)\in H_2(M;\mathbf{Z}/2)$ since both the signature and the fundamental class in $H_4(M)$ are preserved by a homotopy equivalence.

For the line $p+q=5$ in the E_2-term, we have $H_1(M)$ in the $(1,4)$ position, and $H_4(M;\mathbf{Z}/2)\cong\mathbf{Z}/2$ in the $(4,1)$ position. and both these terms survive to E_∞. Under the map $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B)$, the summand $H_1(M)$ maps isomorphically (by Lemma 4.1 again), and the $\mathbf{Z}/2$ summand maps to zero. It follows that $H_2(M;\mathbf{Z}/2)$ lies in the image from $\widehat{\Omega}_5^{Spin}(B,M) \to \widehat{\Omega}_4^{Spin}(M)$, hence $\mathrm{Im}\,\alpha = H_2(M;\mathbf{Z}/2)$. □

Corollary 4.6. $\ker(\widetilde{\mathcal{H}}(M) \to \mathrm{Aut}_\bullet(B)) = \ker(\widetilde{\mathcal{H}}(M) \to \mathrm{Aut}_\bullet(M)) \cong H_1(M)$.

Now we need to compute some homology groups. We need the following special case of a result of P. Teichner.

Lemma 4.7 ([19]). *If M has odd order fundamental group, then $\Gamma(\pi_2(M)) = \mathbf{Z} \oplus \pi_3(M)$ as $\Lambda := \mathbf{Z}[\pi_1(M)]$ modules.*

Proof. Recall that $s_M \in \Gamma(\pi_2)$ denotes the equivariant intersection form on $\pi_2(M)$. The π_1-module $\Gamma(\pi_2)$ sits in the Whitehead sequence

$$0 \to H_4(\widetilde{M}; \mathbf{Z}) \to \Gamma(\pi_2) \to \pi_3(M) \to 0$$

where the first map sends $1 \in H_4(\widetilde{M}; \mathbf{Z})$ to s_M (see [9]). We wish to construct a π_1-homomorphism $f\colon \Gamma(\pi_2) \to \mathbf{Z}$ such that $f(s_M) = 1$. First, consider the map $f_1\colon \Gamma(\pi_2) \to \mathbf{Z}$ given by the composite of the norm $N\colon \Gamma(\pi_2) \to H^0(\pi_1; \Gamma(\pi_2))$ and a map $d\colon H^0(\pi_1; \Gamma(\pi_2)) \to \mathbf{Z}$ chosen so that $d(s_M) = 1$. Such a map exists because s_M is unimodular and is thus a primitive element in $\Gamma(\pi_2)$. We have $f_1(s_M) = |\pi_1|$.

Next, a map $f_2\colon \Gamma(\pi_2) \to \mathbf{Z}$ was constructed in [2], as the composite

$$\Gamma(\pi_2) \subseteq \pi_2 \otimes \pi_2 \cong \mathrm{Hom}(\pi_2^*, \pi_2) \cong \mathrm{Hom}(\pi_2, \pi_2) \to \mathbf{Z}$$

where the middle isomorphisms are defined by $x \otimes y \mapsto (\psi \mapsto \psi(x) \cdot y)$ and $\theta(\psi) := \psi \circ s_M^{-1}$, and the last map is the trace. By definition, $f_2(s_M) = trace(id_{\pi_2}) = \mathrm{rank}\, \pi_2$. But $\mathrm{rank}\, \pi_2 = \chi(\widetilde{M}) - 2 = |\pi_1| \cdot \chi(M) - 2$ is relatively prime to $|\pi_1|$, so we can get a homomorphism $f\colon \Gamma(\pi_2) \to \mathbf{Z}$ with $f(s_M) = 1$ by taking an appropriate linear combination of f_1 and f_2. □

Proposition 4.8. *$H_4(B)$ is torsion-free, and $H_5(B) = 0$.*

Proof. We use the Serre spectral sequence of the fibration $\tilde{B} \to B \to K(\pi, 1)$, E_2-term given by $E_2^{p,q} = H_p(\pi_1; H_q(\tilde{B}))$, where $\pi_1 = \pi_1(M)$, and substitute the values $H_i(\tilde{B}) = 0$, for $i = 1, 3, 5$, $H_2(\tilde{B}) = \pi_2 := \pi_2(M)$, and $H_4(\tilde{B}) = \Gamma(\pi_2)$. We have a splitting $\Gamma(\pi_2) = \mathbf{Z} \oplus \pi_3(M)$ as $\Lambda := \mathbf{Z}[\pi_1]$ modules. But from [2, p. 3] we have $Tors(H_4(B)) \cong \widehat{H}_0(\pi_1, \pi_3(M))$. Also, from [10, §3], and the assumption that π_1 has odd order, we have $\widehat{H}^i(\pi_1; \Gamma(\pi_2)) = \widehat{H}^i(\pi_1; \mathbf{Z})$ in all dimensions. In particular, $\widehat{H}_0(\pi_1, \pi_3(M)) = 0$ implying that $H_4(B)$ is torsion-free, and the term $E_2^{1,4} = H_1(\pi_1; \Gamma(\pi_2)) = H_1(\pi_1; \mathbf{Z})$.

The image of the projection map $H_4(\tilde{B}) \to H_4(B)$ is always a quotient of $E_2^{0,4} = H_0(\pi_1; H_4(\tilde{B}))$ under the edge homomorphism. In our case, $\widehat{H}^{-1}(\pi_1; \mathbf{Z}) = 0$ (for any finite group [3]), so we have an inclusion

$$H_0(\pi_1; \Gamma(\pi_2)) \subset H^0(\pi_1; \Gamma(\pi_2))\ .$$

But $\Gamma(\pi_2)$ is $\mathbf{Z}$-torsion free, hence so is the term $E_2^{0,4} = H_0(\pi_1; H_4(\tilde{B}))$. It follows that this term survives to E_∞ and injects into $H_4(B)$.

Now consider the differentials d_3 in the spectral sequence affecting the lines $p + q = 4, 5$. These have the form $d_3\colon H_{i+3}(\pi_1) \to H_i(\pi_1; \pi_2)$, for $i = 1, 2$ or 3.

We can obtain information about them by comparing the spectral sequence for B with that for B_2, the 2-skeleton of B in some CW-structure. By the results of [9, §2], we have $\pi_2(B_2) = \Omega^3\mathbf{Z}$, and a short exact sequence of stable Λ-modules

$$0 \to \Omega^3\mathbf{Z} \to \pi_2 \to S^3\mathbf{Z} \to 0\,.$$

However, the corresponding differentials in the spectral sequence for B_2 must be isomorphisms (in order that $H_*(B_2) = 0$ for $* > 2$). We can therefore identity our original d_3 differentials with the natural maps

$$H_i(\pi_1; \Omega^3\mathbf{Z}) \to H_i(\pi_1; \pi_2)$$

in the long exact sequence

$$\cdots \to H_{i+1}(\pi_1; S^3\mathbf{Z}) \to H_i(\pi_1; \Omega^3\mathbf{Z}) \to H_i(\pi_1; \pi_2) \to H_i(\pi_1; S^3\mathbf{Z}) \to \dots$$

for the extension describing π_2, by means of the dimension-shifting isomorphism $H_i(\pi_1; \Omega^3\mathbf{Z}) = H_{i+3}(\pi_1)$. Now we compute the maps in this long exact sequence, using the values $H_3(\pi_1; S^3\mathbf{Z}) = \hat{H}_0(\pi_1) = 0$, and $H_2(\pi_1; S^3\mathbf{Z}) = \hat{H}_{-1}(\pi_1) = \mathbf{Z}/|\pi_1|$. Since $H_3(B) = 0$ (following from the fact that $H_3(M) = 0$ and the 3-equivalence $M \to B$), a comparison with the 3-skeleton B_3 shows that the differential $d_3\colon H_4(\pi_1) \to H_1(\pi_1; \pi_2)$ is an isomorphism. We also get the following exact sequences:

$$H_4(\pi_1; \pi_2) \to H_1(\pi_1) \to H_6(\pi_1) \to H_3(\pi_1; \pi_2) \to 0$$

and

$$0 \to H_5(\pi_1) \to H_2(\pi_1; \pi_2) \to \mathbf{Z}/|\pi_1| \to 0$$

determining the other d_3 differentials.

Finally, by comparing to the spectral sequence for the 4-skeleton $B_4 \subset B$ and the spectral sequence for the universal covering $\widetilde{M} \to M$, we can see that the differential $d_3\colon E_2^{4,2} \to E_2^{1,4}$ is just the natural map $H_4(\pi_1, \pi_2) \to H_1(\pi_1)$ above. Furthermore, we can identify the differential $d_5\colon E_3^{6,0} \to E_3^{1,4}$ with the inclusion $\ker(H_6(\pi_1) \to H_3(\pi_1; \pi_2)) \subseteq H_1(\pi_1)$ given by the exact sequence above. This eliminates everything on the line $p + q = 5$, so $H_5(B) = 0$. □

Corollary 4.9. *The group* $\widehat{\Omega}_5^{Spin}(B, M) = H_2(M; \mathbf{Z}/2)$ *and injects into* $\mathrm{Aut}_\bullet(M)$.

Proof. We had a short exact sequence $0 \to H_1(M) \to \Omega_5^{Spin}(B) \to H_5(B)$, but now we know that $H_5(B) = 0$. Therefore

$$\widehat{\Omega}_5^{Spin}(B, M) = \ker(\widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B)),$$

which equals $H_2(M; \mathbf{Z}/2)$. The result now follows by the commutativity of the braid. □

Corollary 4.10. *The images of* $\mathrm{Aut}_\bullet(M)$ *or* $\widetilde{\mathcal{H}}(M)$ *in* $\mathrm{Aut}_\bullet(B)$ *are precisely equal to the isometry group* $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]))$ *of the quadratic 2-type.*

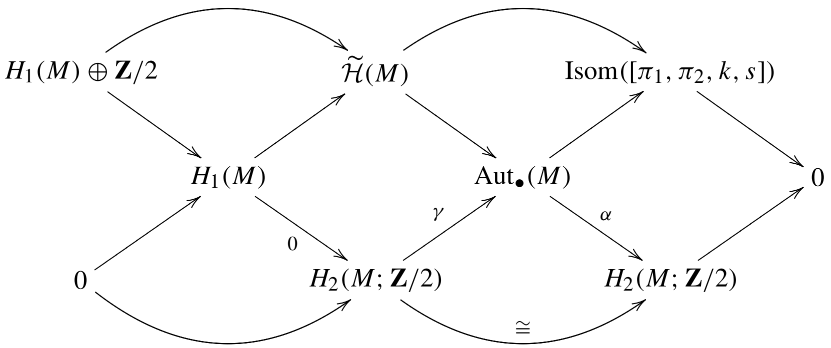
Proof. If $f\colon M \to M$ is an element in $\mathrm{Aut}_\bullet(M)$, then its image in $\Omega_4^{Spin}(B)$ factors through the map $\widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B)$, which has trivial image in $H_4(B)$. Therefore, $c_*(f_*[M]) = c_*[M]$, and since $trf(c_*[M])$ is just the intersection form of M (considered as an element of $H_4(\tilde{B})$ [9, p. 89]), we see that $\mathrm{Im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$ is contained in the isometries of the quadratic 2-type.

However, since $H_4(B)$ is torsion free, it is detected by the transfer map trf : $H_4(B) \to H_4(\tilde{B})$. Now suppose that $\phi\colon B \to B$ is an element of $\mathrm{Aut}_\bullet(B)$ contained in $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]))$. Then

$$trf(\phi_*(c_*[M])) = trf(c_*[M]),$$

and hence $\phi_*(c_*[M]) = c_*[M]$. By [9, 1.3], there exists a lifting $h\colon M \to M$ such that $c \circ h \simeq \phi \circ \phi$. It follows (as in [9, p. 88]) that h is a homotopy equivalence. The result for the image of $\widetilde{\mathcal{H}}(M)$ follows by exactness of the braid, and the fact that $H_4(B)$ is torsion free. □

We can now put the pieces together to establish our main results. Here are the relevant terms of our braid diagram:



The proof of Theorem B. We work in the topological category, and explain the smooth case in Remark 4.12. The first exact sequence (for spin manifolds)

$$1 \to \mathcal{S}(M \times I, \partial) \to \mathcal{H}(M) \to \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]) \to 1$$

is obtained from the diagram in the proof of Lemma 4.1, by replacing $\mathrm{Aut}_\bullet(M)$ with the image of $\widetilde{\mathcal{H}}(M)$ in $\mathrm{Aut}_\bullet(M)$. From the braid diagram, we see that this image is just the isometry group $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$. The exact sequence for $\mathcal{S}(M \times I, \partial)$ is part of the surgery exact sequence [22]. We have just substituted the calculation $L_5^h(\mathbf{Z}[\pi_1(M)]) = 0$ (see [1]), and computed the normal invariant term

$$[M \times I, M \times \partial I; G/TOP] = H^2(M \times I, \partial; \mathbf{Z}/2) \oplus H^4(M \times I, \partial; \mathbf{Z})\ .$$

The fact that $\tilde{L}_6(\mathbf{Z}[\pi_1(M, x_0)])$ injects into $\mathcal{S}^h(M \times I, \partial)$ for odd order fundamental groups is a computation of the surgery obstruction map

$$[M \times D^2, M \times S^1; G/TOP] \to L_6(\mathbf{Z}[\pi_1(M, x_0)])$$

in the surgery exact sequence. This map factors through a bordism group depending functorially on $\pi_1(M, x_0)$ (see [22, Thm. 13B.3]). Since the 2-localization map

$L_*(\mathbf{Z}[\pi]) \to L_*(\mathbf{Z}[\pi]) \otimes \mathbf{Z}_{(2)}$ is an injection for L-groups of finite groups [21, Thm. 7.4], we can use the fact that 2-local bordism is generated by the image from the 2-Sylow subgroup. It follows that the image of the surgery obstruction map $[M \times D^2, \partial; G/TOP] \to L_6(\mathbf{Z}[\pi_1(M, x_0)])$ factors through the 2-Sylow subgroup inclusion $L_6(\mathbf{Z}) \to L_6(\mathbf{Z}[\pi_1(M, x_0)])$. We have given a direct argument here, but this fact about the surgery obstruction map also follows from [8, Thm. A].

In the non-spin case, we must still prove that the image of $\widetilde{\mathcal{H}}(M)$ in $\mathrm{Aut}_\bullet(B)$ is still $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$. This will be done below. □

The proof of Theorem A. In the spin case, the quotient of $\widetilde{\mathcal{H}}(M)$ by the subgroup $H_1(M)$ is isomorphic to $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$. This gives the splitting of the short exact sequence

$$0 \to K \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]) \to 1$$

where $K := \ker(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$. It follows that

$$\mathrm{Aut}_\bullet(M) \cong K \rtimes \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$$

with the conjugation action of $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$ on the normal subgroup K defining the semi-direct product structure. However, the braid diagram also shows that the map γ is an injective *homomorphism*. To check this, first observe that the isomorphism $\widehat{\Omega}_4^{Spin}(M) = H_2(M; \mathbf{Z}/2) \oplus \mathbf{Z}$ is natural, so any self-homotopy equivalence of M which acts as the identity on $H_2(M; \mathbf{Z}/2)$ also acts as the identity on $\widehat{\Omega}_4^{Spin}(M)$. But any element in the image of γ is trivial in $\mathrm{Aut}_\bullet(B)$, so acts as the identity on $H_2(M; \mathbf{Z}/2)$. Then formula (2.1) shows that α is a homomorphism on the image of γ, and a diagram chase using Corollary 4.9 shows that γ is a homomorphism.

Therefore we have a short exact sequence of groups and homomorphisms

$$0 \to H_2(M; \mathbf{Z}/2) \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]) \to 1\ .$$

Moreover, $K = \mathrm{Im}\,\gamma$ and K is mapped isomorphically onto $H_2(M; \mathbf{Z}/2)$ by the map α. Finally, we apply formula (2.1) to obtain the relations:

$$0 = \alpha(id_M) = \alpha(g \circ g^{-1}) = \alpha(g) + g_*(\alpha(g^{-1}))$$

for any $[g] \in \mathrm{Aut}_\bullet(M)$, and

$$\alpha(g \circ f \circ g^{-1}) = g_*(\alpha(f))$$

for any $[f] \in K$. Therefore the conjugation action on K agrees with the induced action on homology under the identification $K \cong H_2(M; \mathbf{Z}/2)$ via α. It follows that

$$\mathrm{Aut}_\bullet(M) \cong H_2(M; \mathbf{Z}/2) \rtimes \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$$

as required, with the action of $\mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$ on the normal subgroup $H_2(M; \mathbf{Z}/2)$ given by the induced action of homotopy self-equivalences on homology. This completes the proof in the spin case.

For the non-spin case we must compute the bordism groups of the normal 2-type. Recall that the first differential in the "James" spectral sequence used to compute $\Omega_*(B\langle w_2\rangle) = \pi_*(M\xi)$ has the same E_2-term as the one used above for $w_2 = 0$, but the differentials are twisted by w_2. In particular, d_2 is the dual of Sq^2_w, where $Sq^2_w(x) := Sq^2(x) + x \cup w_2$.

Proposition 4.11. $\Omega_4(B\langle w_2\rangle) = \mathbf{Z} \oplus \mathbf{Z}/2 \oplus H_4(B)$ *and* $\Omega_5(B\langle w_2\rangle) = H_1(M)$. $\Omega_4(M\langle w_2\rangle) = \mathbf{Z} \oplus H_2(M;\mathbf{Z}/2) \oplus \mathbf{Z}$, *and* $\Omega_5(M\langle w_2\rangle) = H_1(M) \oplus \mathbf{Z}/2$. *The natural map* $\Omega_4(M\langle w_2\rangle) \to \Omega_4(B\langle w_2\rangle)$ *is injective on the* $\mathbf{Z}$ *summands, and is the homomorphism* $w_2\colon H_2(M;\mathbf{Z}/2) \to \mathbf{Z}/2$ *on* $H_2(M;\mathbf{Z}/2)$.

Proof. As before, we only need to compute the d_2 differentials. The point is that the composition

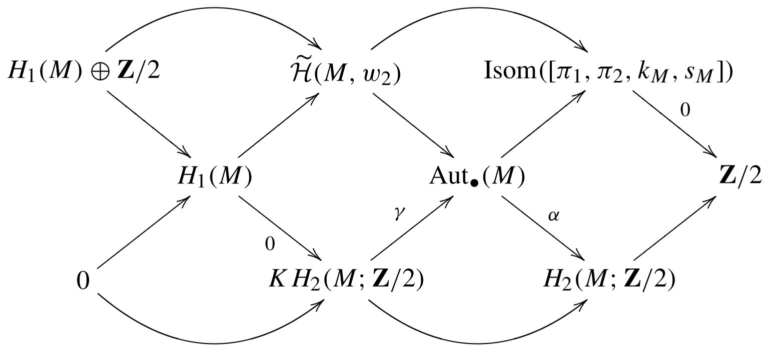
$$H^2(B;\mathbf{Z}/2) \xrightarrow{Sq^2_w} H^4(B;\mathbf{Z}/2) \xrightarrow{Sq^2_w} H^6(B;\mathbf{Z}/2)$$

is exact and the kernel of $Sq^2_w\colon H^2(B;\mathbf{Z}/2) \to H^4(B;\mathbf{Z}/2)$ is the subspace $\langle w_2\rangle \cong \mathbf{Z}/2$. This gives the cokernel $\mathbf{Z}/2$ in the $E^{2,2}_\infty$ position. The same calculation in the spectral sequence for $M\langle w_2\rangle$ uses the fact that

$$Sq^2_w\colon H^2(M;\mathbf{Z}/2) \to H^4(M;\mathbf{Z}/2)$$

is zero, since w_2 is also the first Wu class of M. □

We now continue with the proof of Theorem A and Theorem B in the non-spin case. The relevant terms on our braid are now:



Since the class $w_2 \in H^2(M;\mathbf{Z}/2)$ is a characteristic element for the cup product form (mod 2), it is preserved by the induced map of a self-homotopy equivalence of M. Therefore, the image of $\mathrm{Aut}_\bullet(M)$ in $\Omega_4(M\langle w_2\rangle)$ lies in the subgroup $KH_2(M;\mathbf{Z}/2) := \ker(w_2\colon H^2(M;\mathbf{Z}/2) \to \mathbf{Z}/2)$. It then follows from the braid diagram, that

$$\mathrm{Im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B)) = \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M])$$

just as in the spin case. This completes the proof of Theorem B, and Theorem A follows as in the spin case. □

Remark 4.12. If M and M' are smooth, closed, oriented 4-manifolds, and W is a topological h-cobordism between them, then there is a single obstruction in $H^4(W, \partial W; \mathbf{Z}/2)$ to smoothing W relative to the boundary (see [11, p. 194, 202], or [6, 8.3B]). If $\pi_1(M, x_0)$ has odd order this obstruction vanishes. This implies that the forgetful map $\mathcal{H}_{\mathrm{DIFF}}(M) \to \mathcal{H}_{\mathrm{TOP}}(M)$ is surjective. It is also injective: we compare the smooth and topological surgery exact sequences for $\mathcal{S}^h(M \times I, \partial)$ as in Lemma 4.1, noting that the map on the term $H_1(M; \mathbf{Z})$ is multiplication by 2 (hence an isomorphism). It follows that the calculation in Theorem B also holds for the smooth h-cobordism group $\mathcal{H}_{\mathrm{DIFF}}(M)$. In addition, if M and M' are smooth 4-manifolds which are homeomorphic, then there exists a smooth h-cobordism between them. It follows that the set $\mathcal{H}(M, M')$ of *smooth* h-cobordisms between M and M' is in bijection with $\mathcal{H}_{\mathrm{DIFF}}(M)$, whenever $\mathcal{H}(M, M')$ is non-empty. In particular, $\mathcal{H}(M, M')$ is also computed by Theorem B (extending the result of [14] for the simply-connected case).

References

1. Bak, A.: Odd dimension surgery groups of odd torsion groups vanish. Topology **14**, 367–374 (1975)
2. Bauer, S.: The homotopy type of a 4-manifold with finite fundamental group, Algebraic topology and transformation groups (Göttingen, 1987), Lecture Notes in Math., vol. 1361, Springer, Berlin, 1988, pp. 1–6
3. Brown, K. S.: Cohomology of groups, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original
4. Cochran, T. D., Habegger, N.: On the homotopy theory of simply connected four manifolds. Topology **29**, 419–440 (1990)
5. Dold, A., Whitney, H.: Classification of oriented sphere bundles over a 4-complex. Ann. of Math. (2) **69**, 667–677 (1959)
6. Freedman, M. H., Quinn, F.: Topology of 4-manifolds. Princeton University Press, Princeton, NJ, 1990
7. Gottlieb, D. H.: A certain subgroup of the fundamental group. Amer. J. Math. **87**, 840–856 (1965)
8. Hambleton, I., Milgram, R. J., Taylor, L., Williams, B.: Surgery with finite fundamental group. Proc. London Math. Soc. (3) **56** (2), 349–379 (1988)
9. Hambleton, I., Kreck, M.: On the classification of topological 4-manifolds with finite fundamental group. Math. Ann. **280**, 85–104 (1988)
10. Hambleton, I., Kreck, M.: Cancellation, elliptic surfaces and the topology of certain four-manifolds. J. Reine Angew. Math. **444**, 79–100 (1993)
11. Kirby, R. C., Siebenmann, L. C.: Foundational essays on topological manifolds, smoothings, and triangulations. Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88
12. Kreck, M.: Surgery and duality. Ann. of Math. (2) **149** (3), 707–754 (1999)
13. Kreck, M.: A guide to the classification of manifolds. Surveys on surgery theory. Vol. 1, Princeton Univ. Press, Princeton, NJ, 2000, pp. 121–134
14. Kreck, M.: h-cobordisms between 1-connected 4-manifolds. Geom. Topol. **5**, 1–6 (2001)
15. Lashof, R.: Poincaré duality and cobordism. Trans. Amer. Math. Soc. **109**, 257–277 (1963)

16. Lawson, T.: *h*-cobordisms between simply connected 4-manifolds. Topology Appl. **28**, 75–82 (1988)
17. Spanier, E. H.: Algebraic topology. McGraw-Hill Book Co., New York, 1966
18. Stong, R. E.: Notes on cobordism theory. Mathematical notes, Princeton University Press, Princeton, N.J., 1968
19. Teichner, P.: Topological four manifolds with finite fundamental group. Ph.D. Thesis, University of Mainz, 1992
20. Teichner, P.: On the signature of four-manifolds with universal covering spin. Math. Ann. **295**, 745–759 (1993)
21. Wall, C. T. C.: On the classification of Hermitian forms. V. Global rings, Invent. Math. **23**, 261–288 (1974)
22. Wall, C. T. C.: Surgery on compact manifolds, second ed., American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A. A. Ranicki